Indian Statistical Institute, Bangalore
B. Math (II)
Second semester 2003-2004
Semestral Examination : Statistics (II)

Date: 03-05-2004 **Maximum Score 80** **Duration: 3 Hours**

1. Suppose there are 12 train reservation counters in the city of Bangalore. A counter is open on any particular day with probability θ, $0 < \theta < 1$. The counters function independent of each other. For reasons of proximity and convenience Shilpak uses either counter 1 or counter 2. Shilpak is interested in knowing *i)* $\psi_1(\theta)$, the probability that either counter 1 is open or counter 2 is open on any given day and *ii)* $\psi_2(\theta)$, the probability that exactly one of the counters 1 and 2 is open on a given day. Let $X_i = 1(0)$ if the *ith* counter is open (closed) on a given day, $1 \leq i \leq 12$. Let $X_1,\ X_2, ..., X_{12}$ be the random sample taken on some day indicating whether various of the reservation counters are open or not.

   (a) Find $\psi_1(\theta)$ and $\psi_2(\theta)$.

   (b) Show that $T = \sum_{i=1}^{12} X_i$ is a minimal sufficient statistic for θ.

   (c) Is $T = \sum_{i=1}^{12} X_i$ complete as well? Substantiate.

   (d) Find Fisher information $I(\theta)$ contained in the sample $X_1,\ X_2, ..., X_{12}$ about θ.

   (e) Find an unbiased estimator for $\psi_2(\theta)$. Hence or otherwise obtain *UMVUE* for $\psi_2(\theta)$.

   [2+3+4+3+6 = 18]

2. Suppose that an electronic system contains n similar components which function independently of each other and which are connected in series, so that the system fails as soon as one of the components fails. Suppose $X_1,\ X_2, ..., X_n$ denote the lifetimes of the n components. Suppose also that the lifetime of each component, measured in hours, has *exponential distribution* with *pdf* $\lambda e^{-\lambda x} I_{(0,\infty)}(x)$. The system user has reasons to believe that λ has a *prior distribution* given by *Gamma*(a, b), $a > 0$, $b > 0$ known.

   (a) Find the distribution of Y, the lifetime of the system. Determine $E(Y)$ $(= \frac{1}{\theta}$ say), the expected lifetime of the system. Let

   (b) Obtain *posterior distribution* of θ given the observation Y. Obtain mean and variance of the *posterior distribution* of θ.

   (c) Suggest *Bayes estimator* for θ.

   [3+8+1 = 12]

3. Let $X_n, Y_n, n \geq 1$ be sequences of random variables and X be a random variable, all defined on the same probability space, such that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$, where c is a finite constant. Prove that $X_n + Y_n \xrightarrow{d} X + c$.

   [08]

4. Let $X_1,\ X_2, ..., X_n$ be a random sample from $N(\mu, \sigma^2)$ with both the mean μ and the variance σ^2 being unknown. Derive a level α *likelihood ratio test (LRT)* for testing the hypotheses $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$.

   [08]

[PTO]

5. The manufacturer of a certain type of automobile claims that under typical urban driving conditions the automobile will travel at least 20 *km* per litre of petrol. The owner of an automobile of this type notes the mileages that she obtained in her own urban driving conditions when she fills the tank with petrol on 9 different occasions. She finds that the results *km per litre*, on different occasions were as follows :

   15.6, 18.6, 18.3, 20.1, 21.5, 18.4, 19.1, 20.4, 19.0.

   (a) List carefully the assumptions you must make and formulate the problem of testing of hypotheses to ascertain the manufacturer's claim.
   (b) Carry out a test at 5% level of significance.
   (c) Find the *p-value* of the test.
   (d) Find 10% *confidence interval* for the expected distance travelled per litre of petrol

   [3+6+2+3 = 14]

6. Let $X_1, X_2, ..., X_n$ be a random sample from $N(\theta, 1)$, the mean θ being unknown. Consider the problem of testing the hypotheses $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$

   (a) Show that the family of densities of $\overline{X}$ possesses *monotone likelihood ratio (MLR) property.*
   (b) Show that the power function $\beta(\theta)$ of the test that rejects H_0 if $\overline{X} \geq c$ is increasing in θ, where $c = \frac{1}{\sqrt{n}} z_{(1-\alpha)} + \theta_0$ and z_p is the *p th quantile* of $N(0, 1)$ distribution.
   (c) Obtain the *size* of the above test in *(b)* and show that the test is *unbiased.*
   (d) **Derive** a level α *likelihood ratio test (LRT)* for testing the hypotheses $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$.
   (e) Are the tests in (b) and (d) the same?

   [2+2+3+6+1 = 14]

7. Oxide layers on semiconductor wafers are etched in a mixture of gases to achieve the proper thickness. The variability in the thickness of these oxide layers is a critical characteristic of the wafer, and a low variability is desirable for subsequent processing steps. Two different mixtures of gases are being studied to determine whether one is superior in reducing the variability of the oxide thickness. Sixteen wafers are etched in each gas. The following data was collected $\sum_{i=1}^{16} \left(X_i - \overline{X}\right)^2 = 57.06$ and $\sum_{i=1}^{12} \left(Y_i - \overline{Y}\right)^2 = 68.1$.

   (a) List carefully the assumptions you must make before formulating the problem.
   (b) Formulate and carry out the problem of testing of hypotheses to check whether there is any evidence to indicate that either gas is preferable at $\alpha = .05$?
   (c) Find p-value.

   [2+6+2 = 10]